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ON A METHOD FOR THE UNIFORMIZATION OF SOLUTIONS IN CENTRAL MOTION PROBLEMS*

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A method is proposed within the framework of linear and regular celestial mechanics /1/, which at the expense of introducing regularizing variables permits the removal of a pole-type singularity existing in the presence of a central body and which also reduces the equations of motion to linear form by giving them the form of the equations of motion of a harmonic oscillator. This connection with the theory of oscillations of a harmonic oscillator permits an analysis from a single viewpoint of various types of motions because the energy constant h occurs as a parameter in the equation itself. A relation determining the regularizing function under a specified field potential is obtained. Using regularization the solutions can be represented in a uniformized form, which avoids the necessity of examining the branching of the solutions, arising when going around the critical points. Uniformization in the large /2/ is achieved by using elliptic functions. The perturbed Kepler motion is considered as an application of the uniformization method.

The regularization of equations of motion plays an important role in celestial mechanics, especially in the analysis of the collision of bodies. Its seems regularization was first suggested by Euler /3/ for a one-dimensional problem of collison of two bodies. In /4/ it was shown that Euler's setting of the problem leads to the regularization of the restricted three-body problem as well. A regularization allowing the two-body problem to be reduced to the harmonic oscillator problem in the complex plane was shown in /5/. Subsequent progress in this area occurs in /6,7/ in which spinor regularization is proposed, being a generalization of the Levi-Civita regularization /5/ for the three-dimensional two-body problem. In recent years a large number of papers have appeared dealing mainly with various aspects of the Kepler problem (see /8/, for instance). In this connection the regularization methods proved particularly effective for the analysis of perturbed Kepler motions. Questions on the regularization of the equations of motions for conservative systems with n degrees of freedom were examined, in particular, in /9,10/.

1. Basic relations. Let the representative point M of a system with a reduced mass μ move in a central force field with a potential $U(r)$ under an energy constant h. Introducing the polar coordinates r and φ in the orbit plane (the origin $r=0$ is combined with the central body) and using the energy integral $^1/_2\,\mu v^2 + U(r) = h$ and the area integral $r^2 d\varphi/dt = c$ (c is the area constant), we obtain, after the elimination of φ

$$\frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 = h - U_1(r) \quad \left(U_1(r) = U(r) + \frac{\mu c^2}{2r^2}\right) \tag{1.1}$$

Here $U_1(r)$ is the reduced potential energy. A qualitative analysis of Eq.(1.1) for a prescribed form of potential energy $U(r)$ can be made by the Weierstrass method /11/. Below we indicate another method, based on the introduction of a regularizing transformation of time. The idea for the method goes back to /12/.

2. Regularization of time. We introduce a new regularizing independent variable $\tau = \tau(t)$ by setting

$$d\tau = g^{-1}(r)\,dt \quad (g(r) > 0) \tag{2.1}$$

Here $g(r)$ is a function of variable r, belonging to class C^1 and not vanishing in the phase space domain corresponding the original system's motion. In accord with transformation (2.1), Eq.(1.1), after differentiation with respect to τ and reduction by the nonzero factor $dr/d\tau$ takes the form

$$\mu\frac{d^2r}{d\tau^2} = \frac{d}{dr}\left(g^2(r)\,(h - U_1(r))\right) \tag{2.2}$$

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To linearize the equation obtained we must require that the right-hand side of (2.2) be a linear function of r. Hence as a result of integration we obtain the fundamental relation for determining the regularizing function $g(r)$ for a prescribed form of the total potential $U_1(r)$

$$g^2(r)(h - U_1(r)) = {}^1/_2 c_1 r^2 + c_2 r + c_3 \quad (c_1,\ c_2,\ c_3 = \text{const}) \tag{2.3}$$

Expression (2.3) can be looked upon also as a relation for defining the form of the total potential $U_1(r)$, admitting of the regularization being examined under a prescribed from of the regularizing function $g(r)$. In accordance with (2.2) and (2.3) we obtain a linear equation written as an equation of motion of a harmonic oscillator

$$d^2r/d\tau^2 = c_1 r / \mu + c_2 / \mu \tag{2.4}$$

The values $r = r_1$ and $r = r_2$ for which the radial velocity $v_r = dr/dt$ vanishes are the roots of the equation $h - U_1(r) = 0$. Consequently, representing the right-hand side of (2.3) as $B(r - r_1) \times (r - r_2)\ (B = \text{const})$, we obtain the following relations between the parameters:

$$c_1 = 2B,\ c_2 = -B(r_1 + r_2),\ c_3 = Br_1r_2 \tag{2.5}$$

Note that in the case of real values r_1 and r_2 the discriminant $\Delta = c_2^2 - 2c_1c_3 \geqslant 0$.

3. Uniformization of solutions. When integrating (2.4) we should distinguish the cases: $c_2/c_1 > 0$, $c_2/c_1 < 0$ and $c_2/c_1 = 0$.

a) Let $c_2/c_1 > 0$. In this case the general integral of (2.4) is

$$r = A \operatorname{ch}(\nu\tau + \alpha) - c_2/c_1 = a(e \operatorname{ch} s - 1),\quad a = c_2/c_1,\ A = ae > 0,\ \nu^2 = c_1/\mu,\ s = \nu\tau + \alpha \tag{3.1}$$

Here A and α are integration constants, a is called the mean distance, and e is the quasi-eccentricity. To determine the latter we find the roots of the equation $dr/ds = 0$ in the form $r_1 = a(e - 1)$ and $r_2 = -a(e + 1)$, which, according to (2.5), leads to the relation

$$e^2 = 1 - 2c_1^{-1}c_3a^{-2} = 1 - 2c_1c_3c_2^{-2} \tag{3.2}$$

To determine the true anomaly $\varphi^* = \varphi + \text{const}$ we make use of the area integral. Introducing the regularizing variable τ of (2.1) and directing the abscissa axis x along the apsidal line in the direction toward the perigee $\varphi^* = \varphi$), with due regard to (3.1) we obtain

$$d\varphi = \frac{2c}{\nu a^2} \frac{(r^*)^{-3/2} g(ar^*)\, dr^*}{\sqrt{4r^*((r^* + 1)^2 - e^2)}} \quad \left(r^* = \frac{r}{a}\right) \tag{3.3}$$

We introduce a new variable $y = r^* - m$, where the parameter m is chosen from the condition, the terms quadratic in y in radicand (3.3) vanish. This condition yields $m = -^2/_3$ and equality (3.3) becomes

$$d\varphi = \frac{2c}{\nu a^2} \frac{(y + m)^{-3/2} g(a(y + m))\, dy}{\sqrt{4(y - e_1)(y - e_2)(y - e_3)}} \tag{3.4}$$

Here the roots $e_i\ (i = 1,\ 2,\ 3)$ satisfy the conditions $e_1 > e_2 > e_3$, $e_1 + e_2 + e_3 = 0$, while their values are determined by the expressions

$$e > 1;\ e_1 = e - {}^1/_3,\ e_2 = {}^2/_3,\ e_3 = -(e + {}^1/_3),\quad e < 1;\ e_1 = {}^2/_3,\ e_2 = e - {}^1/_3,\ e_3 = -(e + {}^1/_3) \tag{3.5}$$

For the integration of (3.4) we introduce the Weierstrass elliptic function, setting $y = \wp(u;\ g_2,\ g_3)$, where according to (3.5) the invariants g_2 and g_3 are

$$g_2 = -4(e_1e_2 + e_2e_3 + e_3e_1) = 4(1 + 3e^2)/3,\quad g_3 = 4e_1e_2e_3 = 8(1 - 9e^2)/27 \tag{3.6}$$

Taking advantage of the relation

$$\wp'^2(u) = 4(\wp(u) - e_1)(\wp(u) - e_2)(\wp(u) - e_3) \quad (' = d/du)$$

well known from the theory of elliptic functions /13/, as a result of integrating (3.4) we obtain

$$\varphi = \frac{2c}{\nu a^2} \int (\wp(u) + m)^{-3/2} g(a(\wp(u) + m))\, du + \text{const} \tag{3.7}$$

We use (2.1) and (3.1) to determine the time t. We have

$$t = 2\nu^{-1}\int (\wp(u) + m)^{1/2} g(a(\wp(u) + m))\,du + \text{const} \tag{3.8}$$

Adding on to (3.7) and (3.8) the expression

$$r = a(\wp(u; g_2, g_3) + m) \tag{3.9}$$

for r, we obtain the complete solution of the problem in uniformized form, where u plays the role of the uniformized variable.

b) Let $c_2/c_1 < 0$. In this case the general integral of (2.4) is

$$r = A \operatorname{ch}(\nu\tau + \alpha) - c_2/c_1 = a(e \operatorname{ch} s + 1)$$

$$a = |c_2/c_1|, \; A = ae > 0, \; \nu^2 = c_1/\mu, \; s = \nu\tau + \alpha \tag{3.10}$$

Here A and α are integration constants. Analogously to case a), omitting the intermediate computations, we obtain the uniformized solution as

$$\begin{aligned} r &= a(\wp(u; g_2^*, g_3^*) + m^*) \\ \varphi &= 2c\nu^{-1}a^{-2}\int (\wp(u) + m^*)^{-3/2} g(a(\wp(u) + m^*))\,du + \text{const} \\ t &= 2\nu^{-1}\int (\wp(u) + m^*)^{1/2} g(a(\wp(u) + m^*))\,du + \text{const} \end{aligned} \tag{3.11}$$

Here u is the uniformizing variable, $m^* = -m = 2/3$, and the roots e_1^*, e_2^*, e_3^*, respectively, are

$$e > 1;\; e_1^* = -e_3,\; e_2^* = -e_2,\; e_3^* = -e_1, \quad e < 1;\; e_1^* = -e_3,\; e_2^* = -e_2,\; e_3^* = -e_1 \tag{3.12}$$

The relations $g_2^* = g_2$ and $g_3^* = -g_3$ are valid for the invariants g_2^* and g_3^*.

c) Let $c_2/c_1 = 0$. In this case the general integral of (2.4) is

$$r = A \operatorname{ch}(\nu\tau + \alpha) = a \operatorname{ch} s \quad (A = a) \tag{3.13}$$

Here A and α are integration constants. Using the uniformization method, we obtain the solution as

$$\begin{aligned} r &= a\wp(u; g_2, g_3) \\ \varphi &= 2c\nu^{-1}a^{-2}\int \wp^{-3/2}(u)\, g(a\wp(u))\,du + \text{const} \\ t &= 2\nu^{-1}\int \wp^{1/2}(u)\, g(a\wp(u))\,du + \text{const} \\ e_1 &= 1,\; e_2 = 0,\; e_3 = -1,\; g_2 = 4,\; g_3 = 0 \end{aligned} \tag{3.14}$$

4. Generalized Sundman transformation.

The generalized Sundman transformation /12/ is of special interest when the regularizing function is a power of r, i.e., $g(r) = r^n$ $(n \neq 0)$. According to (2.1), we have

$$dt = r^n d\tau \tag{4.1}$$

When $n = 1$ we obtain the Sundman transformation $dt = r d\tau$. A transformation of form (4.1) was examined in /14/. The reduced potential $U_1(r)$ admitting of the regularization being examined, under a specified magnitude of the energy constant h and a specified form of the regularizing function $g(r) = r^n$, has, by virtue of the fundamental relation (2.3), the form

$$U_1(r) = h - r^{-2n}(1/2\, c_1 r^2 + c_2 r + c_3) \tag{4.2}$$

Setting $g(r) = r^n = a^n r^{*n}$ in the formulas of section 3, we obtain the corresponding uniformized solutions.

a) $c_2/c_1 > 0$. According to (3.7)—(3.9) the solution is

$$\begin{aligned} r &= a(\wp(u; g_2, g_3) + m) \\ \varphi &= 2c\nu^{-1}a^{n-2}\int (\wp(u) + m)^{n-3/2}\,du + \text{const} \\ t &= 2\nu^{-1}a^n\int (\wp(u) + m)^{n+1/2}\,du + \text{const} \end{aligned} \tag{4.3}$$

Here $m = -{}^2/_3 = -e_2$ when $e > 1$ and $m = -{}^2/_3 = -e_1$ when $e < 1$. The invariants g_2 and g_3 are determined by (3.6).

b) $c_2/c_1 < 0$. Here, according to (3.11) the solution has the same form as in the preceding case of $c_2/c_1 > 0$ if m, g_2 and g_3 are replaced by $m^* = -m$, $g_2^* = g_2$ and $g_3^* = -g_3$, respectively.

c) $c_2/c_1 = 0$. According to (3.14) the solution is

$$
\begin{aligned}
r &= a\wp(u;\, g_2,\, g_3) \\
\varphi &= 2c\nu^{-1}a^{n-2}\int \wp^{n-3/2}(u)\, du + \text{const} \\
t &= 2\nu^{-1}a^{n}\int \wp^{n+1/2}(u)\, du + \text{const}
\end{aligned}
\tag{4.4}
$$

As follows from the formulas presented, the uniformization of the solutions is achieved with the aid of elliptic functions for the values $n = {}^1/_2, {}^3/_2, 5/2, \ldots$

Examples. 1. Let $n = {}^1/_2$. The regularizing transformation is

$$d\tau = r^{-1/2}\, dt \quad (g(r) = r^{1/2})$$

The total potential $U_1(r)$ admitting of the regularization indicated, to (4.2) and allowing for the equality $c_2 = h$, is

$$U_1(r) = -{}^1/_2\, c_1 r - c_3/r$$

Let us consider the case when $c_2/c_1 > 0$ and $e > 1$. Here $m = -{}^2/_3 = -e_2$ and $(e_2 - e_1)(e_2 - e_3) = 1 - e^2$. Introducing the Weierstrass zeta-function $\zeta(u)$ and using the relation /13/

$$\wp(u + \omega_1 + \omega_3) - e_2 = \frac{(e_2 - e_1)(e_2 - e_3)}{\wp(u) - e_2}$$

where ω_1 and ω_3 are the half-periods of function $\wp(u)$, the integration of (4.3) yields the solution

$$
\begin{aligned}
r &= a(\wp(u;\, g_2,\, g_3) - e_2) \\
\varphi &= 2c\nu^{-1}a^{-3/2}(e^2 - 1)^{-1}(\zeta(u + \omega_1 + \omega_3) + e_2 u) + \text{const} \\
t &= 2\nu^{-1}a^{1/2}(-\zeta(u) - e_2 u) + \text{const}
\end{aligned}
$$

The solutions when $c_2/c_1 < 0$ and $c_2/c_1 = 0$ are obtained analogously.

2°. Let $n = {}^3/_2$. The regularizing transformation (2.1) becomes

$$d\tau = r^{-3/2}dt \quad (g(r) = r^{3/2}) \tag{4.5}$$

The total potential $U_1(r)$ admitting of the regularization being examined, according to (4.2), is found to be

$$U_1(r) = -\frac{{}^1/_2 c_1}{r} - \frac{c_2}{r^2} - \frac{c_3}{r^3} \tag{4.6}$$

The energy constant h equals zero.

a) $c_2/c_1 > 0$. From (4.3), with due regard to the expressions

$$\int \wp(u)\, du = -\zeta(u), \quad \int \wp^2(u)\, du = \frac{1}{6}\wp'(u) + \frac{1}{12}g_2 u$$

as a result of integration we obtain the solution

$$
\begin{aligned}
r &= a(\wp(u;\, g_2,\, g_3) + m) \\
\varphi &= 2c\nu^{-1}a^{-1/2}\, u + \text{const} \\
t &= 2\nu^{-1}a^{3/2}\left(\frac{1}{6}\wp'(u) - 2m\zeta(u) + \left(\frac{1}{12}g_2 + m^2\right)u\right) + \text{const}
\end{aligned}
\tag{4.7}
$$

Thus, the trajectory equation for the motion of parabolic type $(h = 0)$ being examined has the form

$$r = a\left(\wp\left(\frac{\nu a^{1/2}}{2c}\,\varphi + \alpha\right) + m\right) \quad (\alpha = \text{const}) \tag{4.8}$$

b) $c_2/c_1 < 0$. Here the solution has the same form as in the preceding case with the replacement in formulas (4.7) and (4.8) of parameter m by $m^* = -m$ and of invariants g_2 and g_3 by $g_2^* = g_2$ and $g_3^* = -g_3$, respectively.

c) $c_2/c_1 = 0$. Here the solution has the same form as in case a) if in formulas (4.7) and (4.8) we set $m = 0$ and take the invariants as $g_2 = 4$ and $g_3 = 0$.

5. Perturbed Kepler motion. We consider a perturbed Kepler motion in a central field with potential

$$U(r) = -\frac{a_1}{r} - \frac{a_2}{r^2} - \frac{a_3}{r^3} \tag{5.1}$$

In particular, the first approximation potential of an oblate spheroid for points moving in the equatorial plane $(a_2 = 0)$ /15,16/, as well as the third approximation potential energy of an atomic field /17/, have such a form. The correction to the Newtonian potential $(-a_1/r)$ in the form of an additive summand $(-a_3/r^3)$ can be analyzed as well in the general theory of relativity to explain the perihelion motion of Mercury /18/.

From (5.1) and (2.3) it follows that the regularizing function must be chosen as

$$g(r) = r^{3/2}(1 + \beta r)^{-1/2} \tag{5.2}$$

where parameter β is subject to later determination. For this choice of the regularizing function $g(r)$ the following relations between the parameters

$$2h = \beta c_1, \ 2a_1 = c_1 + 2\beta c_2, \ a_2 - {}^1/_2\, \mu c^2 = c_2 + \beta c_3, \ a_3 = c_3 \tag{5.3}$$

are valid. Eliminating c_1, c_2 and c_3 from these relations, we obtain a cubic equation

$$a_3\beta^3 - (a_2 - {}^1/_2\, \mu c^2)\beta^2 + a_1\beta - h = 0 \tag{5.4}$$

for the determination of β. Equation (5.4) always has at least one real root. When all three roots β_1, β_2 and β_3 are real, when choosing the value of β we should allow for the condition $g(r) > 0$ which is fulfilled when $\beta > 0$, while for $\beta < 0$ the domain of the motions being examined is bounded by a circle of radius $r < |\beta|^{-1}$. In the general case, to determine the domain of possible motions we can make use of the Hill curves /19/.

Let us now turn the consideration of the different types of motions.

1°. Case of motion of hyperbolic type $(h > 0)$. Without loss of generality we set $\beta > 0$. Then $c_1 = 2h\beta^{-1} > 0$ and, consequently, the sign of c_2/c_1 depends on the sign of $c_2 = (a_1\beta - h)\beta^{-2}$.

a) Let $0 < h < a_1\beta$. This yields $c_2/c_1 > 0$ and, consequently, we can use (3.1). As a result we obtain $r = a(e \operatorname{ch} s - 1)$, where the mean distance a and the quasi-eccentricity e, according to (3.2) and (5.3), are

$$a = (a_1\beta - h)/(2h\beta), \quad e^2 = 1 - \beta a_3/(ha^2) \tag{5.5}$$

Hence, in particular, it follows that $a_3 < 0$ necessarily for the fulfillment of $e > 1$ (when $h > 0$). To determine the true anomaly φ we use (3.3) wherein $g(r)$ is determined by (5.2). We introduce a new variable $y = 1/r - m$, choosing the parameter m from the condition that the terms quadratic in y in radicand (3.3) vanish. This yields

$$m = -\frac{1}{3}(a\beta + \alpha_1 - \alpha_2) \quad (\alpha_1 = (e + 1)^{-1}, \quad \alpha_2 = (e - 1)^{-1}) \tag{5.6}$$

and, finally, we obtain

$$d\varphi = -\frac{c\sqrt{2a\mu}}{\sqrt{a_3}} \frac{dy}{\sqrt{4y^3 - g_2 y - g_3}} \tag{5.7}$$

$$g_2 = 4(3m^2 - a\beta(\alpha_1 - \alpha_2) + \alpha_1\alpha_2)$$
$$g_3 = -4(m + a\beta)(m + \alpha_1)(m - \alpha_2) \tag{5.8}$$

Uniformization will be achieved if we set $y = \wp(u; g_2, g_3)$. As a result we find

$$\varphi = -c(2a\mu)^{1/2}(a_3)^{-1/2}u + \text{const}, \quad a/r = \wp(u; g_2, g_3) + m \tag{5.9}$$

The trajectory equation is

$$a/r = \wp(k\varphi + \alpha) + m \quad (k = (2a\mu)^{-1/2}c^{-1}a_3^{1/2}, \quad \alpha = \text{const}) \tag{5.10}$$

b) Let $a_1\beta < h < \infty$, which corresponds to the case $c_2/c_1 < 0$. The solution has the same form as in the preceding case if α_1 and α_2 are replaced, respectively, by $\alpha_1^* = -\alpha_1$ and $\alpha_2^* = -\alpha_2$.

c) Let $h = a_1\beta$, which corresponds to the case $c_2/c_1 = 0$. Making use of (3.13) and noting that $r_1 r_2 = -a^2$ in the case at hand, in accord with (2.5) and (5.3) we obtain

$$a^2 = |a_3|\,\beta/h \quad (a_3 < 0)$$

The solution itself will have the same form as in the case of $c_2/c_1 > 0$ if we set $\alpha_1 = \alpha_2 = 1$ in the expressions (5.6) and (5.8) for m, g_2 and g_3.

2°. Case of motion of elliptic type $(h < 0)$. Here the solution will have the same form as in the case $c_2/c_1 > 0$ if we replace α_1 and α_2 by $\alpha_1^* = -\alpha_1$ and $\alpha_2^* = -\alpha_2$, respectively, and note that $e < 1$ and $a_3 > 0$ for the case being examined.

3°. Case of motion of parabolic type $(h = 0)$. Here the value $\beta = 0$ is one of the solutions of (5.4) and, consequently, the regularizing transformation (5.2) reduces to the generalized Sundman transformation (4.5). Thus, the solution will be obtained if in the formulas corresponding to case $n = 3/2$, according (4.6), (5.1) and (5.3) we set

$$c_1 = 2a_1, \quad c_2 = a_2 - {}^1/_2\mu c^2, \quad c_3 = a_3$$

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